

## GRADUATE SEMINAR SPRING 2007 PROBLEM SET 1

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### 1. PROBLEM 1

Show that for  $a_i > 0, p_i > 0$

$$(1.1) \quad \lim_{n \rightarrow \infty} \sqrt[n]{p_1 a_1^n + p_2 a_2^n + \cdots + p_l a_l^n} = \max a_i$$

First, suppose that  $a_1$  and  $a_2$  are the maximum elements of  $a_i$ . That means that  $a_1 = a_2$  and therefore we can write  $p_1 a_1^n + p_2 a_2^n = \tilde{p}_1 a_1^n$  where  $\tilde{p}_1 = p_1 + p_2$ , and we now have a set of  $l - 1$  distinct constants which have a unique maximum. Obviously, if the maximum is shared by  $k$  constants then we can redefine our  $p_i$ 's such that we have a set of  $l - k$  distinct constants. We may therefore consider the case where  $a_i$  has a unique maximum. Let  $\max a_i = a_k$  with  $1 \leq k \leq l$ .

We now wish to pull out the largest term inside the square root,  $a_k$ . For a fixed  $n$  we have

$$(1.2) \quad a_k \sqrt[n]{p_1} \leq a_k \sqrt[n]{p_1 \left(\frac{a_1}{a_k}\right)^n + p_2 \left(\frac{a_2}{a_k}\right)^n + \cdots + p_k + \cdots + p_l \left(\frac{a_l}{a_k}\right)^n} \leq a_k \sqrt[n]{p_1 + p_2 + \cdots + p_l}$$

Using the fact that  $\lim_{n \rightarrow \infty} a^{1/n} = a^0 = 1$  for  $a > 0$ , we can take the limit of the inequality as  $n \rightarrow \infty$ , which gives us an upper and lower bound of  $a_k$ . Thus, by the Squeezing Theorem, we have that

$$(1.3) \quad \lim_{n \rightarrow \infty} \sqrt[n]{p_1 a_1^n + p_2 a_2^n + \cdots + p_l a_l^n} = a_k = \max a_i$$

□

### 2. PROBLEM 2

Show that for  $a_i > 0, p_i > 0$

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{p_1 a_1^{n+1} + p_2 a_2^{n+1} + \cdots + p_l a_l^{n+1}}{p_1 a_1^n + p_2 a_2^n + \cdots + p_l a_l^n} = \max a_i$$

If there is more than one maximum of  $a_i$  we may proceed as in the previous problem by redefining the  $p_i$ 's, so it is sufficient to consider only the case of a  $a_i$  having a unique maximum. Let  $\max a_i = a_k$  with  $1 \leq k \leq l$ .

Similar to the previous problem, we will pull out the largest factors of the numerator and denominator,  $a_k$ . For a fixed  $n$  we have

$$(2.2) \quad s_n = \frac{a_k^{n+1} \left( p_1 \left( \frac{a_1}{a_k} \right)^{n+1} + p_2 \left( \frac{a_2}{a_k} \right)^{n+1} + \cdots + p_k + \cdots + p_l \left( \frac{a_l}{a_k} \right)^{n+1} \right)}{a_k^n \left( p_1 \left( \frac{a_1}{a_k} \right)^n + p_2 \left( \frac{a_2}{a_k} \right)^n + \cdots + p_k + \cdots + p_l \left( \frac{a_l}{a_k} \right)^n \right)}$$

Since  $\frac{a_i}{a_k} < 1$  for all  $i \neq k$ , in taking the limit we find

$$(2.3) \quad \lim_{n \rightarrow \infty} s_n = \frac{a_k^{n+1} p_k}{a_k^n p_k} = a_k = \max a_i$$

□

### 3. PROBLEM 3

Let  $f(x)$  be a polynomial whose zeros are all real and positive and for which

$$(3.1) \quad -\frac{f'(x)}{f(x)} = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots$$

Show that

$$(3.2) \quad \lim_{k \rightarrow \infty} \frac{1}{\sqrt[k]{c_k}} = \lim_{k \rightarrow \infty} \frac{c_{k-1}}{c_k}$$

exists and that it is equal to the smallest zero of  $f(x)$ .

Let  $f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ . We recognize the left hand side of (3.1) as the logarithmic derivative, so we have

$$(3.3a) \quad -\frac{f'(x)}{f(x)} = -\frac{d}{dx} (\log f(x))$$

$$(3.3b) \quad = -\frac{d}{dx} \left( \sum_{i=1}^n \log(x - \alpha_i) \right)$$

$$(3.3c) \quad = -\sum_{i=1}^n \frac{1}{x - \alpha_i}$$

$$(3.3d) \quad = \sum_{i=1}^n \frac{1}{\alpha_i - x}$$

$$(3.3e) \quad = \sum_{i=1}^n \frac{1}{\alpha_i \left( 1 - \frac{x}{\alpha_i} \right)}$$

$$(3.3f) \quad = \sum_{i=1}^n \frac{1}{\alpha_i} \left( 1 + \frac{x}{\alpha_i} + \left( \frac{x}{\alpha_i} \right)^2 + \cdots \right)$$

$$(3.3g) \quad = \sum_{i=1}^n \frac{1}{\alpha_i} + x \sum_{i=1}^n \left( \frac{1}{\alpha_i} \right)^2 + \cdots + x^k \sum_{i=1}^n \left( \frac{1}{\alpha_i} \right)^{k+1}$$

We have now found the constants  $c_k$  which are given by

$$(3.4) \quad c_k = \sum_{i=1}^n \left( \frac{1}{\alpha_i} \right)^{k+1}$$

Let  $\alpha_j$  denote the smallest of the  $\alpha_i$ . Since all of the roots  $\alpha_i$  are positive, all of the terms in the sum are positive, so we easily find a lower bound

$$(3.5) \quad \sqrt[k]{\left(\frac{1}{a_j}\right)^{k+1}} = \frac{1}{\alpha_j} \sqrt[k]{\frac{1}{\alpha_j}} \leq \sqrt[k]{c_k}$$

Since  $\alpha_j$  is the smallest root, the largest term in the sum is  $\left(\frac{1}{\alpha_j}\right)^{k+1}$ . We can then find an upper bound as

$$(3.6) \quad \sqrt[k]{c_k} \leq \sqrt[k]{n \left(\frac{1}{\alpha_j}\right)^{k+1}} = \frac{1}{\alpha_j} \sqrt[k]{\frac{n}{\alpha_j}}$$

Putting these results together we have

$$(3.7) \quad \frac{\alpha_j}{\sqrt[k]{\frac{1}{\alpha_j}}} \leq \frac{1}{\sqrt[k]{c_k}} \leq \frac{\alpha_j}{\sqrt[k]{\frac{n}{\alpha_j}}}$$

In the limit as  $k \rightarrow \infty$  the denominators converge to 1 and therefore by the Squeeze theorem  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[k]{c_k}} = \alpha_j$ , the smallest root of  $f(x)$ . As for the showing that  $\lim_{n \rightarrow \infty} \frac{c_{k-1}}{c_k} = \alpha_j$  one may use exactly the same method as in Problem 2, which is also a ratio of finite polynomials.  $\square$

#### 4. PROBLEM 4

Let  $f(x)$  and  $\phi(x)$  be positive, continuous functions on  $[a, b]$ . Show that

$$(4.1) \quad \lim_{n \rightarrow \infty} \sqrt[n]{\int_a^b \phi(x) (f(x))^n dx}$$

exists and is equal to the maximum of  $f(x)$  on  $[a, b]$ .

Let us assume that  $f(x)$  attains a single maximum at  $x = \zeta$  in the interval  $[a, b]$ , denoted by  $\max_{[a, b]} f(x) = f(\zeta)$ . We can now write the integral with the maximum value of the integrand "factored out":

$$(4.2) \quad f(\zeta)^n \sqrt[n]{\int_a^b \phi(x) \left(\frac{f(x)}{f(\zeta)}\right)^n dx}$$

If we can bound this from above and below by sequences that converge to  $f(\zeta)$ , then the Squeeze Theorem can be used. Since  $\left(\frac{f(x)}{f(\zeta)}\right)^n < 1$  we can easily find an upper bound that converges to  $f(\zeta)$

$$(4.3) \quad f(\zeta)^n \sqrt[n]{\int_a^b \phi(x) \left(\frac{f(x)}{f(\zeta)}\right)^n dx} \leq f(\zeta)^n \sqrt[n]{\int_a^b \phi(x) dx}$$

To get a lower bound takes a bit more work. Because  $f(x)$  is continuous we know that for any  $\epsilon > 0$  there exists a  $\delta(\epsilon)$  such that

$$(4.4) \quad f(\zeta) - \epsilon \leq f(x) \leq f(\zeta) + \epsilon$$

for  $\zeta - \delta \leq x \leq \zeta + \delta$ . Using this we can find a lower bound

$$(4.5) \quad f(\zeta) \sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x) \left(\frac{f(\zeta) - \epsilon}{f(\zeta)}\right)^n dx} \leq f(\zeta) \sqrt[n]{\int_a^b \phi(x) \left(\frac{f(x)}{f(\zeta)}\right)^n dx}$$

Since everything inside the integrand is constant except  $\phi(x)$  we have

$$(4.6) \quad f(\zeta) \sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x) \left(\frac{f(\zeta) - \epsilon}{f(\zeta)}\right)^n dx} = (f(\zeta) - \epsilon) \sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x) dx}$$

Now  $\sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x) dx}$  is just a positive number which converges to 1 as  $n \rightarrow \infty$ . We now have

$$(4.7) \quad f(\zeta) - \epsilon \leq \lim_{n \rightarrow \infty} f(\zeta) \sqrt[n]{\int_a^b \phi(x) \left(\frac{f(x)}{f(\zeta)}\right)^n dx} \leq f(\zeta)$$

Now let  $\epsilon \rightarrow 0$  and we have the required result.  $\square$